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CALCULATION OF THE FORCE AND MOMENT OF FORCES ACTING ON A DROP IN AN ARBITRARY NON-STEADY FLOW OF A VISCOUS FLUID*

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The Oseen point force method /1-3/ which differs from the methods used earlier in similar problems, is used to obtain formulas for the force and moment of forces acting on a spherical drop in an inhomogeneous non-steady flow of viscous incompressible fluid. In special cases the results can be reduced to well-known results.

Earlier, the non-steady motion of a rigid particle in an inhomogeneous non-steady flow was considered in /4, 5/, its rotation in /6, 7/, the conditions of slippage at the surface in /5, 7/, and the effect of the specified external forces in /8, 9/. The corresponding stationary problem was studied in /10, 11/ and the non-steady motion of a drop in uniform non-steady flow in /12–14/.

1. Formulation of the problem. A liquid sphere of viscosity μ', density ρ' and constant radius a, moves with velocity $\mathbf{u}(t)$ through an incompressible medium of viscosity μ and density ρ. The problem is studied in the Stokes approximation, i.e. we consider the following linear, non-steady equations of motion of the fluid outside and inside the drop:

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$$\rho\partial\mathbf{v}/\partial t = -\nabla p + \mu\Delta\mathbf{v} + \mathbf{f}_w, \quad \operatorname{div}\mathbf{v} = 0 \tag{1.1}$$

$$\rho'\partial\mathbf{v}'/\partial t = -\nabla p' + \mu'\Delta\mathbf{v}' + \mathbf{f}_w', \quad \operatorname{div}\mathbf{v}' = 0 \tag{1.2}$$

When there is no particles, the fluid has a velocity $\mathbf{v}_\infty(\mathbf{r}, t)$ and a pressure $p_\infty(\mathbf{r}, t)$, which also satisfy the system of Eqs.(1.1).

The conditions satisfied at the drop surface are the conditions that no fluid passes across the surface of contact and that the tangential components of the velocity are equal to the tangential components of the viscous stress tensor

$$v_n = v_n' = u_n, \quad \mathbf{v}_\tau = \mathbf{v}_\tau', \quad \mathbf{p}_{n\tau} = \mathbf{p}_{n\tau}' \tag{1.3}$$

The following conditions hold away from the particle and at its centre:

$$r\to\infty, \quad \mathbf{v}-\mathbf{v}_\infty\to 0, \quad p-p_\infty\to 0; \quad r=0, \quad |\mathbf{v}'|\neq\infty, \quad p'\neq\infty$$

Let us introduce the dimensionless variables by dividing the radius vector, the time, velocity, pressure and the given external force by the quantities $a, a^2/\nu, \nu/a, \mu\nu/a^2, \mu\nu/a^3$ $(\mu = \rho\cdot\nu)$ respectively.

In the outer problem we can eliminate the given force $\mathbf{f}_w(\mathbf{r}, t)$ by subtracting the equations for $\mathbf{v}_\infty, p_\infty$ from (1.1). We will assume that for the inner problem $\mathbf{f}_w' = \nabla\varphi$ (e.g. the force of gravity), and this can then be included in the pressure. Since the deviation of the drop shape from spherical (i.e. the conditions for p_{nn}) were not considered, it follows that the boundary conditions in this case remain unchanged.

As a result, we obtain the following equations and boundary conditions in terms of the dimensionless variables $\mathbf{w} = \mathbf{v} - \mathbf{v}_\infty$, $q = p - p_\infty$, v', p':

$$\partial\mathbf{w}/\partial t = -\nabla q + \Delta\mathbf{w}, \ \operatorname{div}\mathbf{w} = 0 \tag{1.4}$$

$$\sigma B^{-2}\,\partial\mathbf{v}'/\partial t = -\nabla p' + \sigma\Delta\mathbf{v}', \ \operatorname{div}\mathbf{v}' = 0 \tag{1.5}$$

$$r = 1, \quad w_n + v_{\infty n} = v_n' = u_n, \quad \mathbf{w}_\tau + \mathbf{v}_{\infty\tau} = \mathbf{v}_\tau', \quad \mathbf{q}_{n\tau} + \mathbf{p}_{\infty n\tau} = \mathbf{p}_{n\tau}' \tag{1.6}$$

$$r\to\infty, \ \mathbf{w}\to 0, \ q\to 0 \tag{1.7}$$

$$r = 0, \ |\mathbf{v}'|\neq\infty, \ p'\neq\infty \tag{1.8}$$

The system of Eqs.(1.4) and (1.5), which contains two dimensionless parameters σ and B^2 equal to the ratios of the dynamic and kinematic coefficients inside and outside the drop respectively, together with conditions (1.6)-(1.8), enables us to determine the velocity and pressure field outside and inside the drop.

2. Constructing the solutions. We shall use the Laplace transform with parameter s, with respect to the argument of time t. Here the equations of motion take the form

$$s\mathbf{W} = -\nabla Q + \Delta\mathbf{W}, \ \operatorname{div}\mathbf{W} = 0 \tag{2.1}$$

$$\sigma B^{-2}s\mathbf{V}' = -\nabla P' + \sigma\Delta\mathbf{V}', \ \operatorname{div}\mathbf{V}' = 0 \tag{2.2}$$

$$r = 1, \quad W_n + V_{\infty n} = V_n' = U_n, \quad \mathbf{W}_\tau + \mathbf{V}_{\infty\tau} = \mathbf{V}_\tau', \quad \mathbf{Q}_{n\tau} + \mathbf{P}_{\infty n\tau} = \mathbf{P}_{n\tau}' \tag{2.3}$$

$$r\to\infty, \ \mathbf{W}\to 0, \ Q\to 0 \tag{2.4}$$

$$r = 0, \ |\mathbf{V}'|\neq\infty, \ P'\neq\infty \tag{2.5}$$

Following the Oseen point force method /1, 2/, we shall construct the solutions of the above system of equations with help of the fundamental tensor u_{ki} and vector p_k

$$u_{ki} = \delta_{ki}\Delta\Phi - \frac{\partial^2\Phi}{\partial x_k\partial x_i}, \quad p_k = \frac{\partial}{\partial x_k}(s\Phi - \Delta\Phi)$$

The function Φ is a solution of the equation

$$\Delta\Delta\Phi - s\Delta\Phi = \delta(\mathbf{r}), \quad \Phi = (1 - e^{-\sqrt{s}r})(4\pi sr)^{-1}$$

In this case, when k is fixed the quantities u_{ki} and p_k satisfy the equations

$$s\mathbf{u}_k = -\nabla p_k + \Delta\mathbf{u}_k, \quad \operatorname{div}\mathbf{u}_k = 0$$

The solutions $F_k^1\mathbf{u}_k$ and $F_k^1 p_k$ ($\mathbf{F}^1$ is a constant vector) correspond to the flow generated by a point force situated at the origin of coordinates. Clearly, any order derivatives with respect to the coordinations of $\mathbf{u}_k$ and p_k will be solutions of system (2.1). In particular, we can say this of the quantities v_{ki}, q_k

$$v_{ki} = \Delta u_{ki} = \delta_{ki}\Delta\Delta\Phi - \partial^2\Delta\Phi/\partial x_k\partial x_i, \quad q_k = \Delta p_k = 0 \quad (r\geqslant 1)$$

In order to satisfy the boundary conditions, we shall construct the outer solution of problem (2.1)-(2.5) with help of the derivatives with respect to the coordinates of the quantities u_{ki}, v_{ki} and p_k. Collecting the terms accompanying $\Delta\Phi$ and separating in the corresponding tensors the symmetric and antisymmetric parts, we obtain expressions for the velocity and pressure field, which in vector notation will be

$$\mathbf{W} = \boldsymbol{\Lambda}\Delta\Delta\Phi - \nabla(\boldsymbol{\Lambda}\cdot\nabla)\Delta\Phi + [\nabla\times\mathbf{L}]\Delta\Phi - \nabla(\mathbf{F}\cdot\nabla)\left(\frac{1}{4\pi rs}\right), \tag{2.6}$$
$$Q = (\mathbf{F}\cdot\nabla)\left(\frac{1}{4\pi r}\right)$$

The components of the vectors $\mathbf{F}$ and $\mathbf{L}$ are linear differential operators of the form

$$F^i = \sum_{n=1}^{\infty} F^{(n)}_{ijk\ldots l}\frac{\partial^{(n-1)}}{\partial x_j\partial x_k\ldots\partial x_l}, \quad L^i = \sum_{n=1}^{\infty} L^{(n)}_{ijk\ldots l}\frac{\partial^{(n-1)}}{\partial x_j\partial x_k\ldots\partial x_l}$$

We can express the vector $\boldsymbol{\Lambda}$ in the same manner. We will show below that the boundary conditions at the sphere hold, if the tensors $F^{(n)}, L^{(n)}$ and $\Lambda^{(n)}$ are symmetrical over any pair of indices and the contractions over any two indices are equal to zero. The solution (2.6) represents another form of the general Lamb solution /15/. This can be shown by assuming that quqntities of the form $f_n = \Lambda^{(n)}_{ij\ldots k}\, x_i x_j \ldots x_k$ are spherical volume harmonics of degree n, since they are homogeneous with respect to r^n and $\Delta f_n = 0$.

Indeed, we can write the solution (2.6) in the form

$$\mathbf{W} = \sum_{n=1}^{\infty}\Big\{[sn^{-1}\psi_{n-1}(r) - \psi_n(r)]\,\nabla(\Lambda^{(n)}_{ij\ldots k}x_i x_j\ldots x_k) - \tag{2.7}$$
$$r\psi_{n+1}(r)\,\nabla r\,(\Lambda^{(n)}_{ij\ldots k}x_i x_j\ldots x_k) - n^{-1}\psi_n(r)\operatorname{rot}[\mathbf{r}\,(L^{(n)}_{sq\ldots l}x_s x_q\ldots x_l)] -$$
$$\nabla\left[F^{(n)}_{kj\ldots l}\frac{\partial^{(n)}}{\partial x_k\partial x_j\ldots\partial x_l}\left(\frac{1}{4\pi sr}\right)\right]\Big\},\quad Q = \sum_{n=1}^{\infty}F^{(n)}_{kj\ldots l}\frac{\partial^{(n)}}{\partial x_k\partial x_j\ldots\partial x_l}\left(\frac{1}{4\pi r}\right)$$

The function $\psi_n(r) = (r^{-1}d/dr)^n\Delta\Phi$ satisfies the relations

$$\frac{d^2\psi_n}{dr^2} + \frac{2(n+1)}{r}\frac{d\psi_n}{dr} - s\psi_n = 0, \qquad \frac{2n+1}{r^2}\psi_n = \frac{s}{r^2}\psi_{n-1} - \psi_{n+1}$$

For example, we have

$$\psi_0(r) = \Delta\Phi = -\frac{1}{4\pi r}e^{-\sqrt{s}r}, \quad \psi_1(r) = \frac{1+\sqrt{s}r}{4\pi r^3}e^{-\sqrt{s}r},$$
$$\psi_2(r) = -\frac{3+3\sqrt{s}r+sr^2}{4\pi r^5}e^{-\sqrt{s}r}$$

The functions $\psi_n(r)$ are proportional to Bessel functions of fractional order /16/. Remembering also that the tensors $F^{(n)}, L^{(n)}, \Lambda^{(n)}$ are symmetrical over any pair of indices and their contractions over two indices are equal to zero, i.e. that quantities of the form $f_n = \Lambda^{(n)}_{i,j\ldots k}\, x_i x_j \ldots x_k$ are spherical volume harmonics of degree n, we can establish a correspondence between the solution (2.6) and the general Lamb solution /15/.

The solution of system (2.2) inside the drop, constructed in the same manner, has the form

$$\mathbf{V}' = -\nabla(S\cdot\nabla)\Delta\Phi' + S\Delta\Delta\Phi' + [\nabla\times T]\Delta\Phi' +$$
$$\lambda^{-2}\nabla\Big(\sum_{n=1}^{\infty} R^{(n)}_{ij\ldots l}x_i x_j\ldots x_l\Big), \quad P' = -\sigma\sum_{n=1}^{\infty} R^{(n)}_{ij\ldots l}x_i x_j\ldots x_l$$
$$S^i = \sum_{n=1}^{\infty} S^{(n)}_{ij\ldots k}\frac{\partial^{(n-1)}}{\partial x_j\ldots\partial x_k}, \quad T^i = \sum_{n=1}^{\infty} T^{(n)}_{ij\ldots k}\frac{\partial^{(n-1)}}{\partial x_j\ldots\partial x_k}$$

Here the tensors $S^{(n)}, T^{(n)}$ and $R^{(n)}$ are symmetrical over any pair of indices, and their contractions over two indices are equal to zero. The function Φ' represents the solution of the equation

$$\Delta\Delta\Phi' - \lambda^2\Delta\Phi' = 0, \quad \Phi' = \frac{\operatorname{sh}\lambda r}{r}, \quad \lambda^2 = \frac{s}{B^2}, \quad \sigma = \frac{\mu'}{\mu}, \quad B^2 = \frac{\nu'}{\nu}$$

bounded at the point $r = 0$.

We can establish, for the inner solution, a correspondence with the general Lamb solution, just as we did for the outer solution. Below, we shall make use of the stress vector, which can be found using the formula

$$\mathbf{Q}_n = -Q\mathbf{n} + \left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\mathbf{W} + \frac{1}{r}\nabla(\mathbf{r}\cdot\mathbf{W}) = \tag{2.8}$$

$$\sum_{k=1}^{\infty}\{2(k+2)r^{-1}\nabla(b_k r^{-k-1}F^{(k)}_{ij\ldots l}n_i n_j\ldots n_l) -$$
$$s(b_k r^{-k-1}F^{(k)}_{ij\ldots l}n_i n_j\ldots n_l)\,\mathbf{n} - (kr)^{-1}[(k-1)\psi_k(r) +$$
$$r^2\psi_{k+1}(r)]\,\mathrm{rot}\,[\mathbf{r}\,(L^{(k)}_{ij\ldots l}x_i x_j\ldots x_l)] +$$
$$(rk)^{-1}[(2k^2-2+sr^2)\psi_k(r) - 2r^2\psi_{k+1}]\,\nabla\,(\Lambda^{(k)}_{ij\ldots l}x_i x_j\ldots x_l) +$$
$$[2(k+2)\psi_{k+1}(r) - s\psi_k(r)]\,(\Lambda^{(k)}_{ij\ldots l}x_i x_j\ldots x_l)\,\mathbf{n}\},$$
$$b_k = (-1)^k(2k-1)!!/4\pi s$$

We can write the boundary conditions (2.3) in the form /10/

$$W_n + V_{\infty n} = V_n' = U_n, \quad \partial W_n/\partial r + \partial V_{\infty n}/\partial r = \partial V_n'/\partial r \tag{2.9}$$
$$(\mathbf{r}\cdot\mathrm{rot}\,\mathbf{W}) + (\mathbf{r}\cdot\mathrm{rot}\,\mathbf{V}_\infty) = (\mathbf{r}\cdot\mathrm{rot}\,\mathbf{V}'),(\mathbf{r}\cdot\mathrm{rot}\,\mathbf{Q}_n) + (\mathbf{r}\cdot\mathrm{rot}\mathbf{P}_{\infty n}) = (\mathbf{r}\cdot\mathrm{rot}\,\mathbf{P}_n')$$
$$(\mathbf{r}\cdot\mathrm{rot}\,[\mathbf{r}\times\mathbf{Q}_n]) + (\mathbf{r}\cdot\mathrm{rot}\,[\mathbf{r}\times\mathbf{P}_{\infty n}]) = (\mathbf{r}\cdot\mathrm{rot}\,[\mathbf{r}\times\mathbf{P}_n'])$$

We will represent the prescribed velocity field $\mathbf{V}_\infty(\mathbf{r}, s)$ in the form of a Taylor series

$$V_\infty{}^i - U^i = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left[\frac{\partial^{(n-1)}(V^i_\infty - U^i)}{\partial x_j\ldots\partial x_p\partial x_q}\right]_0 x_j\ldots x_p x_q = \sum_{n=1}^{\infty} d^{(n)}_{ij\ldots pq}x_j\ldots x_p x_q$$

The zero index means that the corresponding quantity is taken at the centre of the drop when the latter is not there.

We can expand the field $\mathbf{V}_\infty$ at the drop surface $r=1$ in terms of the independent spherical harmonics as follows:

$$V_{\infty n} - U_n = \sum_{k=1}^{\infty} d^{(k)}_{ij\ldots pq}n_i n_j\ldots n_p n_q = \sum_{k=1}^{\infty} A_k \tag{2.10}$$
$$\frac{\partial V_{\infty n}}{\partial r} = \sum_{k=1}^{\infty}(k-1)\,d^{(k)}_{ij\ldots pq}n_i n_j\ldots n_p n_q = \sum_{k=1}^{\infty} B_k$$
$$-(\mathbf{r}\cdot\mathrm{rot}\,[\mathbf{r}\times\mathbf{P}_{\infty n}]) = \sum_{k=2}^{\infty}(k-1)\,[2(k+1)d^{(k)}_{ij\ldots l}n_i n_j\ldots n_l -$$
$$(k-2)\,d^{(k)}_{ipp\ldots jl}n_i n_j\ldots n_l] = \sum_{k=1}^{\infty} E_k$$
$$(\mathbf{r}\cdot\mathrm{rot}\,\mathbf{V}_\infty) = \sum_{k=2}^{\infty}(k-2)\,\varepsilon_{ilj}d^{(k)}_{ljp\ldots q}n_i n_p\ldots n_q = \sum_{k=1}^{\infty} C_k$$
$$(\mathbf{r}\cdot\mathrm{rot}\,\mathbf{P}_{\infty n}) = \sum_{k=2}^{\infty}(k-2)(k-1)\,\varepsilon_{ilj}d^{(k)}_{ljp\ldots q}n_i n_p\ldots n_q = \sum_{k=1}^{\infty} D_k$$

The quantities A_k, B_k, E_k, C_k, D_k represent spherical surface harmonics of degree k. When $k=1$, the quantities can be easily calculated and used later in determining the force and the moment of forces acting on the drop. For example, we have for A_1, B_1

$$A_1 = (V_\infty{}^i - U^i)\,n_i + 3\sum_{k=1}^{\infty}\frac{[\Delta^k V_\infty{}^i]_0\,n_i}{(2k+3)(2k+1)!},$$
$$B_1 = 3\sum_{k=1}^{\infty}\frac{[\Delta^k V_\infty{}^i]_0\,n_i}{(2k+3)(2k+1)(2k-1)!}$$

The boundary conditions (2.9) are reduced here to

$$(k+1)\,b_k F^{(k)}_{ij\ldots l}n_i n_j\ldots n_l + (k+1)\,\psi_k\Lambda^{(k)}_{ij\ldots l}n_i n_j\ldots n_l + A_k = 0 \tag{2.11}$$
$$kR'^{(k)}_{ij\ldots l}n_i n_j\ldots n_l + \lambda^2(k+1)\,\psi_k' S^{(k)}_{ij\ldots l}n_i n_j\ldots n_l = 0$$
$$(k+1)\,\{-(k+2)\,b_k F^{(k)}_{ij\ldots l}n_i n_j\ldots n_l +$$
$$[(k-1)\,\psi_k + \psi_{k+1}]\,\Lambda^{(k)}_{ij\ldots l}n_i n_j\ldots n_l\} + B_k = k(k-1)\,\lambda^{-2}R'^{(k)}_{ij\ldots l}n_i n_j\ldots n_l +$$
$$(k+1)\,[\psi'_{k+1} + (k-1)\,\psi_k']\,S^{(k)}_{ij\ldots l}n_i n_j\ldots n_l$$
$$2b_k k(k+2)\,F^{(k)}_{ij\ldots l}n_i n_j\ldots n_l + (k+1)^{-1}E_k +$$
$$\{[2(k^2-1)+s]\,\psi_k - 2\psi_{k+1}\}\,\Lambda^{(k)}_{ij\ldots l}n_i n_j\ldots n_l = \sigma\,\{2k(k-1)\,\lambda^{-2}R^{(k)}_{ij\ldots l}n_i n_j\ldots n_l +$$
$$[(2k^2-2+\lambda^2)\,\psi_k' - 2\psi'_{k+1}]\,S^{(k)}_{ij\ldots l}n_i n_j\ldots n_l\}$$

$$
\begin{gathered}
(k+1)\,\psi_k L^{(k)}_{ij\ldots l} n_i n_j \ldots n_l + C_k = (k+1)\,\psi_k' T^{(k)}_{ij\ldots l} n_i n_j \ldots n_l \\
(k+1)\,[(k-1)\,\psi_k + \psi_{k-1}]\, L^{(k)}_{ij\ldots l} n_i n_j \ldots n_l + D_k = \\
\sigma\,(k+1)\,[(k-1)\,\psi_k' + \psi_{k-1}']\, T^{(k)}_{ij\ldots l} n_i n_j \ldots n_l
\end{gathered}
$$

where

$$
\psi_k = \{\psi_k(r)\}_{r=1}, \quad \psi_k' = \{(r^{-1}d/dr)^k \Delta\Phi'\}_{r=1},
$$
$$
R'^{(k)}_{ij\ldots l} = R^{(k)}_{ij\ldots l}\ (k \neq 1), \quad R'^{(1)}_i = R^{(1)}_i - U^i
$$

The functions $\psi_k'(r)$ satisfy relations analogous to the equations for $\psi_k(r)$. The solutions of the system of Eqs. (2.11) are as follows:

$$
\begin{aligned}
&\Lambda^{(k)}_{ij\ldots l} n_i n_j \ldots n_l = \{(k+2)\,[2k\psi_{k+1}' - \sigma\,(2\psi_{k+1}' - \lambda^2\psi_k')]\,A_k - \\
&\quad \psi_{k+1}' E_k - \sigma\,(2\psi_{k+1}' - \lambda^2\psi_k')\,B_k\}\,(s\chi_k)^{-1} \\
&S^{(k)}_{ij\ldots l} n_i n_j \ldots n_l = \{(k+2)\,[2\,(k-1)\,\psi_{k-1} + \psi_k]\,A_k - \\
&\quad \psi_{k-1} E_k + (\psi_k - 2\psi_{k-1})\,B_k\}\,\chi_k^{-1} \\
&F^{(k)}_{ij\ldots l} n_i n_j \ldots n_l = -\,[\psi_k \Lambda^{(k)}_{ij\ldots l} n_i n_j \ldots n_l + (k+1)^{-1} A_k]\,b_k^{-1} \\
&R'^{(k)}_{ij\ldots l} n_i n_j \ldots n_l = -\,(k+1)\,k^{-1}\lambda^2\psi_k' S^{(k)}_{ij\ldots l} n_i n_j \ldots n_l \\
&L^{(k)}_{ij\ldots l} n_i n_j \ldots n_l = \{\psi_k' D_k - \sigma\,[(k-1)\,\psi_k' + \psi_{k+1}']\,C_k\}\,\varphi_k^{-1} \\
&T^{(k)}_{ij\ldots l} n_i n_j \ldots n_l = \{\psi_k D_k - \sigma\,[(k-1)\,\psi_k + \psi_{k+1}]\,C_k\}\,\varphi_k^{-1} \\
&\chi_k(s) = (k+1)\,[\sigma\,(2\psi_{k+1}' - \lambda^2\psi_k')\,\psi_{k-1} + \psi_{k+1}'\,(\psi_k - 2\psi_{k-1})] \\
&\varphi_k(s) = (k+1)\,\{\sigma\,[(k-1)\,\psi_k' + \psi_{k+1}']\,\psi_k - [(k-1)\,\psi_k + \psi_{k+1}]\,\psi_k'\}
\end{aligned}
\tag{2.12}
$$

As we have already said, the quantities A_k, B_k, C_k, D_k, E_k are known, independent spherical surface harmonics of degree k, expressed in terms of the given field $\mathbf{V}_\infty$ by the formulas (2.10). They are easily computed for $k = 1$, and appear in the expressions for the force and moment of forces which are found from the surface integrals containing the stress tensor (2.8) and depending, by virtue of the orthogonal nature of the harmonics, only on the vectors $\mathbf{F}^1$ and $\mathbf{L}^1$ shown in the solutions (2.12).

3. Formulas for the force and moment of forces. The quantities shown are related to the vectors $\mathbf{F}^1$ and $\mathbf{L}^1$ which correspond to a point force and point moment and are determined from Eqs. (2.11) by the relations

$$
\mathbf{G}(s) = \mathbf{F}^1(s) + {}^4/_3\pi s\mathbf{U}(s) - \int \mathbf{F}_w(\mathbf{r}, s)\,d\mathbf{r}
$$
$$
\mathbf{M}(s) = {}^2/_3\,(3 + 3\sqrt{s} + s)\,e^{-\sqrt{s}}\mathbf{L}^1 + s\int [\mathbf{r} \times \mathbf{V}_\infty]\,d\mathbf{r} - \int [\mathbf{r} \times \mathbf{F}_w]\,d\mathbf{r}
$$

Here the integration is carried out over the volume of the unit sphere. Using the inverse Laplace transform, we finally obtain the following expressions for the force $\mathbf{g}(t)$ and moment of forces $\mathbf{m}(t)$ acting on the drop in non-uniform non-steady flow:

$$
\begin{aligned}
\mathbf{g}(t) = {}& 2\pi\,\frac{3\sigma+2}{\sigma+1}\,([\mathbf{v}_\infty]_0 - \mathbf{u}) + 6\int_0^t d\tau\, I_1(t-\tau)\,\frac{d}{d\tau}\,([\mathbf{v}_\infty]_0 - \mathbf{u}) + \\
& \frac{2}{3}\,\pi\,\frac{d}{dt}\,([\mathbf{v}_\infty]_0 - \mathbf{u}) + \frac{4}{3}\,\pi\,\frac{d}{dt}\,[\mathbf{v}_\infty]_0 + \\
& \sum_{k=1}^{\infty} \frac{1}{(2k+1)!}\Big\{2\pi\,\frac{3\sigma+2\,(1-k)}{\sigma+1}\,[\Delta^k\mathbf{v}_\infty]_0 + \frac{6\pi}{2k+3}\,\frac{d}{dt}\,[\Delta^k\mathbf{v}_\infty]_0 + \\
& 6\int_0^t d\tau\,[(1-k)\,I_1(t-\tau) + kI_2(t-\tau)]\,\frac{d}{d\tau}\,[\Delta^k\mathbf{v}_\infty]_0\Big\} - \int \mathbf{f}_w(\mathbf{r}, t)\,d\mathbf{r} \\
\mathbf{m}(t) = {}& \frac{4}{3}\int_0^t d\tau\, I_3(t-\tau)\,\frac{d}{d\tau}\,[\operatorname{rot}\mathbf{v}_\infty]_0 + \frac{4}{15}\,\pi\,\frac{d}{dt}\,[\operatorname{rot}\mathbf{v}_\infty]_0 + \\
& \sum_{k=1}^{\infty} \frac{4}{(2k+3)\,(2k+1)!}\Big\{-2\pi k\,[\operatorname{rot}\Delta^k\mathbf{v}_\infty]_0 + \frac{\pi}{2k+5}\,\frac{d}{dt}\,[\operatorname{rot}\Delta^k\mathbf{v}_\infty]_0 + \\
& \int_0^t d\tau\,[I_3(t-\tau) + 2kI_4(t-\tau)]\,\frac{d}{d\tau}\,[\operatorname{rot}\Delta^k\mathbf{v}_\infty]_0\Big\} - \int [\mathbf{r} \times \mathbf{f}_w]\,d\mathbf{r}
\end{aligned}
$$

The Laplace transforms of the functions $I_i(t)$ are, respectively, $\pi s^{-1}K_i(s)$ $(i = 1, 2, 3, 4)$ and

$$K_1(s) = \xi(s)\left[\sigma\alpha(\lambda) + \beta(\lambda)\right] - (3\sigma + 2)(3\sigma + 3)^{-1},$$
$$K_2(s) = \sigma\alpha(\lambda)\xi(s) - \sigma(\sigma + 1)^{-1}$$
$$K_3(s) = (3 + 3\sqrt{s} + s)\theta(s), \quad K_4(s) = (1 + \sqrt{s})\theta(s)$$
$$\xi(s) = (1 + \sqrt{s})\left[\sigma\alpha(\lambda) + (3 + \sqrt{s})\beta(\lambda)\right]^{-1},$$
$$\theta(s) = \sigma\lambda^2\beta(\lambda)\left[\sigma\lambda^2(1 + \sqrt{s})\beta(\lambda) + (3 + 3\sqrt{s} + s)\gamma(\lambda)\right]^{-1}$$
$$\gamma(\lambda) = \lambda \operatorname{ch}\lambda - \operatorname{sh}\lambda, \quad \beta(\lambda) = \operatorname{sh}\lambda - 3\gamma(\lambda)/\lambda^2,$$
$$\alpha(\lambda) = \gamma(\lambda) - 2\beta(\lambda)$$

When the particle is rigid, we obtain

$$I_1(t) = I_2(t) = \frac{\sqrt{\pi}}{\sqrt{t}}, \quad I_3(t) = \pi\left(3 - e^t \operatorname{erfc}\sqrt{t} + \frac{1}{\sqrt{\pi t}}\right),$$
$$I_4(t) = \pi$$

and in the case of a bubble we have, as $\sigma \to 0$,

$$I_1(t) = {}^4/_3\, \pi e^{9t} \operatorname{erfc}(3\sqrt{t}), \quad I_2(t) = I_3(t) = I_4(t) = 0$$

These two special cases agree with the results obtained in /4-9/. When $\sigma = B = 1$, we can write the following analytic expressions for $I_3(t)$, $I_4(t)$:

$$I_3(t) = \frac{\sqrt{\pi}}{2\sqrt{t}}\left[12t^2 - 6t + 1 - (12t^2 + 6t + 1)\exp\left(-\frac{1}{t}\right)\right]$$
$$I_4(t) = \pi\left[\frac{2t^{3/2}}{\sqrt{\pi}} - \frac{2\sqrt{t}}{\sqrt{\pi}} + \frac{1}{2} - \frac{2t^{3/2}}{\sqrt{\pi}}\exp\left(-\frac{1}{t}\right) - \frac{1}{2}\operatorname{erfc}\left(\frac{1}{\sqrt{t}}\right)\right]$$

For other values of the parameters σ and B the quantities $I_i(t)$ are determined numerially. The results of the computations are shown in Figs.1-6. Curves *1-6* correspond to values of the parameters B^2 equal to 0.001; 0.01; 0.1; 1; 10; 100.

For short times, the following asymptotic formulas hold:

$$I_1(t) = \frac{\sigma\sqrt{\pi}}{(\sigma + B)\sqrt{t}} + \frac{\sigma^2 + 2B^2}{(\sigma + B)^2}\pi - \frac{3\sigma + 2}{3(\sigma + 1)}\pi + \frac{2B[3\sigma B^2(\sigma + B) - (\sigma - 2B)^2]\sqrt{\pi t}}{(\sigma + B)^3} + \ldots$$
$$I_2(t) = \frac{\sigma\sqrt{\pi}}{(\sigma + B)\sqrt{t}} + \frac{\sigma(\sigma - 2B)}{(\sigma + B)^2}\pi + \frac{\sigma}{\sigma + 1}\pi + \frac{6B^2\sigma(B^2 + B\sigma + 2 - \sigma)\sqrt{\pi t}}{(\sigma + B)^3} + \ldots$$
$$I_3(t) = \frac{\sigma\sqrt{\pi}}{(\sigma + B)\sqrt{t}} + \frac{2\pi\sigma}{(\sigma + B)^2}(\sigma - B^2) + \ldots, \quad I_4(t) = \frac{\sigma}{\sigma + B}\pi - \frac{4\sigma B(B + 1)}{(\sigma + B)^2}\sqrt{\pi t} + \ldots$$

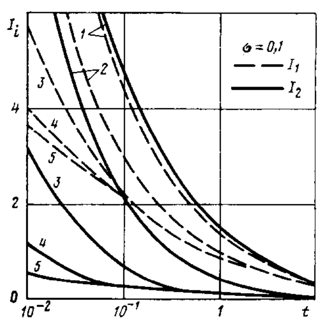

Fig.1

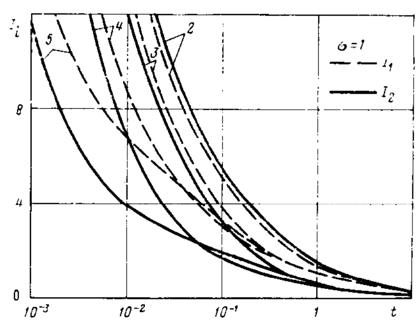

Fig.2

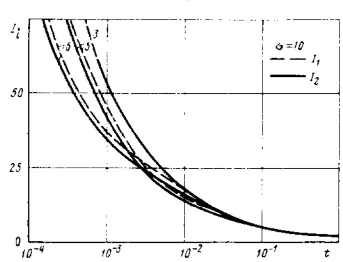

Fig.3

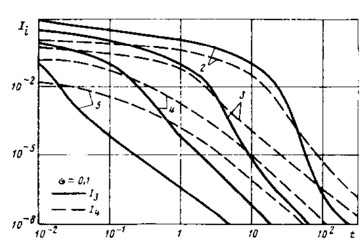

Fig.4

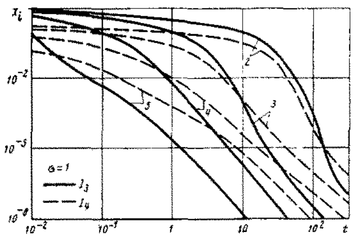

Fig.5

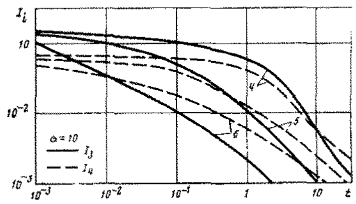

Fig.6

As $t \to \infty$, the functions $I_i(t)$ tend to

$$I_1(t) = \frac{(2+3\sigma)^2 \sqrt{\pi}}{9(1+\sigma)^2 \sqrt{t}} - \frac{2+3\sigma}{54(1+\sigma)^3}\left[\frac{2\sigma}{7B^2} + \frac{2+3\sigma}{3(1+\sigma)}\right]\frac{\sqrt{\pi}}{t^{3/2}} + \ldots,$$

$$I_2(t) = \frac{\sigma(2+3\sigma)\sqrt{\pi}}{3(1+\sigma)^2 \sqrt{t}} - \frac{\sigma}{18(1+\sigma)^3}\left[\frac{4\sigma+2}{7B^2} + \frac{2+3\sigma}{3(1+\sigma)}\right]\frac{\sqrt{\pi}}{t^{3/2}} + \ldots,$$

$$I_3(t) = \frac{\sigma^2}{120B^4}\frac{\sqrt{\pi}}{t^{7/2}} + \ldots$$

$$I_4(t) = \frac{\sigma}{60B^2}\frac{\sqrt{\pi}}{t^{5/2}} + \ldots$$

The ends of the curves in Figs.1-6 are described by the asymptotic formulas with an error not exceeding 10—15%.

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